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Harmonic Superpositions of Non-extremal p -branesI.V. Lavrinenko[‡], H. Lü[†], C.N. Pope^{‡1} and T.A. Tran[‡][†]*Laboratoire de Physique Théorique de l'École Normale Supérieure*²*24 Rue Lhomond - 75231 Paris CEDEX 05*[‡]*Center for Theoretical Physics, Texas A&M University, College Station, Texas 77843***ABSTRACT**

The plot of allowed p and D values for p -brane solitons in D -dimensional supergravity is the same whether the solitons are extremal or non-extremal. One of the useful tools for relating different points on the plot is vertical dimensional reduction, which is possible if periodic arrays of p -brane solitons can be constructed. This is straightforward for extremal p -branes, since the no-force condition allows arbitrary multi-centre solutions to be constructed in terms of a general harmonic function on the transverse space. This has also been shown to be possible in the special case of non-extremal black holes in $D = 4$ arrayed along an axis. In this paper, we extend previous results to include multi-scalar black holes, and dyonic black holes. We also consider their oxidation to higher dimensions, and we discuss general procedures for constructing the solutions, and studying their symmetries.

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1 Introduction

M-theory and string theories admit a plethora of solitonic extended-object solutions. These include both extremal p -branes, which satisfy a BPS saturation condition, and non-extremal, or “black” p -branes. Various ways of enumerating and classifying the BPS-saturated states in toroidally-compactified theories have been developed. One of these involves filling out a web of interconnections between the various points in a plot of p versus spacetime dimension D , by making use of two different dimensional-reduction procedures. The first of these, which is known as diagonal dimensional reduction, involves performing a Kaluza-Klein compactification of one or more of the spatial p -brane dimensions, thereby reducing p and D simultaneously [1, 2]. The second procedure, known as vertical dimensional reduction, instead involves performing the Kaluza-Klein reduction on one or more of the dimensions in the transverse space [3, 4, 5]. This lowers D , while keeping p fixed. In this latter reduction scheme, it is necessary first to construct a suitable configuration of p -branes in the higher dimension that is independent of the transverse coordinates for which the compactification is to be performed. For extremal p -branes this is quite straightforward since arbitrary multi-centre solutions can be constructed, owing to the existence of a no-force condition between individual extremal p -branes [6]. Thus one can make configurations in the higher dimension in which the centres are distributed uniformly over a line, plane or hyperplane, which can then be compactified by the usual Kaluza-Klein procedure.

It is of interest to try to achieve a similar web of interconnections for the case of non-extremal p -brane solitons. It has been shown that there is a universal prescription for “blackening” any extremal p -brane to give a non-extremal one [7, 8], and so the plot of the allowed values of p versus D for such solutions will be the same as in the extremal case. What at first sight is lacking, however, is a fully analogous set of reduction procedures for spanning the two-dimensional (p, D) plane. Actually there is no difficulty in achieving the trajectories of the diagonal dimensional reduction, since no features specific to extremal solutions were used in this case. However, for the vertical reduction it was important that one should be able to construct the necessary multi-centre solutions in the higher dimension, and it is commonly held that the zero-force condition implied by extremality is essential for this. In fact, as was argued in [9, 10], this is not really the case. The point here is that the ability to construct equilibrium multi-centre solutions with arbitrary positions for the centres is an unnecessary luxury, if all one wants to do is to build p -brane configurations that are uniformly distributed over a line, plane or hyperplane. It is in fact sufficient to be able to construct multi-centre solutions where the centres are aligned in an infinite

periodic array, along the line, plane or hyperplane. In any such configuration, there will always be a net balance of forces on each individual p -brane, and so the system will be in equilibrium, albeit an unstable one. However, this instability need not be an obstacle to the explicit construction of the periodic solutions. In fact, by imposing the periodicity on the compactifying coordinates, even the instability is eliminated. In [10], such multi-centre solutions for dilatonic black holes in $D = 4$ were obtained, generalising previous results for black holes in Einstein gravity [11] and the Maxwell-Einstein system [12]. A generalisation to periodic multi-centre p -branes in $(p + 4)$ dimensions was also given in [10]; these solutions can be obtained by performing a diagonal oxidation of the multi-centre black holes in $D = 4$. Although the physical argument given above indicates that periodic multi-centre generalisations should exist for *any* non-extremal p -brane, the complexity of the equations in cases where $p < D - 4$ seems to be too great to allow explicit solutions to be found.

In this paper, we shall investigate some broader classes of multi-centre non-extremal p -branes, although again restricted to the $D = p + 4$ trajectory. We begin by studying multi-scalar non-extremal black holes in $D = 4$. These correspond to configurations that are supported by a number of independently-specifiable charges carried by different 2-form field strengths, reducing to single-scalar black holes if the charges are set equal. Then, we generalise these solutions to multi-scalar black p -branes in $(p + 4)$ dimensions. Another case that would be of great interest to study is the dyonic black hole in four dimensions [13]. This is unusual in that, viewed as a bound state of its two constituents, one with electric and the other with magnetic charge, it has negative binding energy [13] even when extremal. In this paper we are able to obtain some special multi-centre solutions for the dyonic black holes, but unfortunately the general multi-centre solutions with separated electric and magnetic charges seem to be too complicated to construct explicitly. We do, however, present some general discussion on approaches to solving the equation, and also of solution-generating symmetry groups.

2 Multi-charge multi-centre black holes

In this section, we shall consider black-hole solutions in four-dimensional supergravity. Thus the relevant part of the four-dimensional Lagrangian will be of the form

$$\mathcal{L} = eR - \frac{1}{2}e(\partial\vec{\phi})^2 - \frac{1}{4}e \sum_{\alpha=1}^N e^{\vec{a}_\alpha \cdot \vec{\phi}} F_2^{(\alpha)^2}, \quad (2.1)$$

where $\vec{\phi}$ denotes the set of dilatonic scalar fields, $F_2^{(\alpha)}$ are N 2-form field strengths, and \vec{a}_α are a set of constant vectors describing the couplings of the dilatonic scalars to the field strengths. In the context of toroidal compactifications of M theory, the expressions for \vec{a}_α for all the field strengths may be found in [14]. However, we should emphasise that our discussion of multi-centre solutions is more general, and need not necessarily be related to any supergravity theory. We shall be looking at solutions where the black holes carry independent charges for the various field strengths $F^{(\alpha)}$, implying that a corresponding number of scalar fields are non-vanishing.

Since we are looking for solutions describing an array of black holes aligned along an axis, we may assume the following axially-symmetric form for the metric ansatz:

$$ds^2 = -e^{2U} dt^2 + e^{2K-2U} (dr^2 + dz^2) + e^{-2U} r^2 d\theta^2, \quad (2.2)$$

where U and K are both functions of r and z only. (Actually, the most general axially-symmetric ansatz, after simplification by choosing coordinates appropriately, would have an independent function e^{2B} rather than e^{-2U} in the final term. However, it can be shown that after making use of the equations of motion, one may choose coordinates such that $B = -U$ [15].) In this section, we shall consider the case where all the field strengths carry electric charges, and thus their potentials may be taken to be of the purely electric form

$$A^{(\alpha)} = \gamma_\alpha dt, \quad (2.3)$$

where the functions γ_α again depend on r and z only. The dual configurations where the field strengths carry purely magnetic charges instead are, as usual, very similar in structure. The final solutions for the metrics will be identical, and the solutions for the dilatonic scalars will differ only by a sign.

The multi-scalar solutions that are of interest to us here may be found by using the techniques described in [16]. For a set of field strengths and scalars with couplings described by a generic set of vectors \vec{a}_α in (2.1), the equations of motion cannot be decoupled, and as far as we know they would not be solvable explicitly. However, in the supergravity theories obtained by the toroidal reduction of M -theory, there are sets of vectors which, in the special case of 2-form field strengths in $D = 4$, satisfy the relations [14]

$$\vec{a}_\alpha \cdot \vec{a}_\beta = 4\delta_{\alpha\beta} - 1. \quad (2.4)$$

This is precisely the condition that allows the system of equations of motion following from (2.1) to be diagonalised.

To solve the equations, it is convenient to define $\varphi_\alpha \equiv \vec{a}_\alpha \cdot \vec{\phi}_\alpha$, and to introduce new variables Φ_α and Y in place of φ_α and U ;

$$\Phi_\alpha = -\varphi_\alpha + 2U , \quad Y = 2U - \frac{1}{4} \sum_{\alpha=1}^N \Phi_\alpha , \quad (2.5)$$

where we are considering the case in which N field strengths $F^{(\alpha)}$ carry charges. It is straightforward to show that the equations of motion following from (2.1), with the ansätze (2.2) and (2.3), become

$$\begin{aligned} \nabla^2 Y &= 0 , & \nabla^2 \Phi_\alpha &= 2 e^{\Phi_\alpha} (\nabla \gamma_\alpha)^2 , \\ \nabla^2 \gamma_\alpha &= \nabla \Phi_\alpha \cdot \nabla \gamma_\alpha . \end{aligned} \quad (2.6)$$

Note that the Laplacian operator here is that of flat 3-dimensional space in cylindrical polar coordinates, acting on axially-symmetric functions, namely

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{d^2}{dz^2} , \quad (2.7)$$

and $\nabla f \cdot \nabla g = f' g' + \dot{f} \dot{g}$, where a prime denotes a derivative with respect to r , and a dot denotes a derivative with respect to z .

Later, in section 5, we shall discuss rather general procedures for solving the equations. For now, we shall simply present the solutions that are relevant to our discussion. As in the previously studied cases of axially-symmetric solutions to the Einstein [15, 11], Einstein-Maxwell [12] and Einstein-Maxwell-Dilaton [10] systems, we find here that a broad class of solutions can be obtained in terms of axially-symmetric harmonic functions in the (r, θ, z) space. Specifically, we find that the equations of motion are satisfied if

$$\begin{aligned} \nabla^2 Y &= 0 , \\ e^{-\frac{1}{2}\Phi_\alpha} &= \left(e^{-\tilde{U}_\alpha} - c_\alpha^2 e^{\tilde{U}_\alpha} \right) \\ \gamma_\alpha &= c_\alpha e^{2\tilde{U}_\alpha} \left(1 - c_\alpha^2 e^{2\tilde{U}_\alpha} \right)^{-1} , \\ K' &= -\frac{r}{4-N} (\dot{Y}^2 - Y'^2) - \frac{1}{4} r \sum_\alpha (\dot{\tilde{U}}_\alpha^2 - \tilde{U}'_\alpha{}^2) , \\ \dot{K} &= \frac{2r}{4-N} \dot{Y} Y' + \frac{1}{2} r \sum_\alpha \dot{\tilde{U}}_\alpha \tilde{U}'_\alpha , \end{aligned} \quad (2.8)$$

where the c_α are constants. (Note that if $N = 4$ the matrix of dot products (2.4) has vanishing determinant, implying that Y vanishes.) The functions \tilde{U}_α are arbitrary harmonic functions, satisfying

$$\nabla^2 \tilde{U}_\alpha = 0 . \quad (2.9)$$

The final two equations in (2.8) are automatically consistent with the solutions for the other functions, in the sense that the z derivative of the former is equal to the r derivative of the latter, by virtue of the harmonicity of Y and \tilde{U}_α . Thus the solution for K is reduced to quadratures, and the entire solution is given in terms of the $N + 1$ independent arbitrary harmonic functions Y and \tilde{U}_α . The parameters c_α characterise the charges carried by each of the field strengths $F_2^{(\alpha)}$, with $c_\alpha = 0$ corresponding to a zero charge. We shall explain this in more detail below.

The results presented above describe rather broad classes of axially-symmetric solutions to the equations of motion following from (2.1). A subset of these will correspond to the configurations that we are seeking to construct, namely sets of multi-charge black holes arrayed along the z axis. In order to identify which are the solutions that describe these configurations, we need first to be able to recognise the basic single-centre non-extremal multi-charged black hole in this coordinate system. In the previous non-scalar or single-scalar cases, it turned out that the required harmonic function corresponding to the single black hole configuration was precisely that describing the Newtonian gravitational potential of a uniform thin rod of mass k and length $\frac{1}{2}k$ [11, 12, 10]. We find in this case that it is again this same harmonic function that describes the single-centre multi-charge black hole. Specifically, we should take

$$\tilde{U}_\alpha = \tilde{U} \equiv \frac{1}{2} \log \frac{\sigma + \tilde{\sigma} - k}{\sigma + \tilde{\sigma} + k}, \quad Y = \frac{1}{2}(4 - N) \tilde{U}, \quad (2.10)$$

where $\sigma = \sqrt{r^2 + (z - k/2)^2}$ and $\tilde{\sigma} = \sqrt{r^2 + (z + k/2)^2}$, implying that the function U appearing in the metric (2.2) is given by

$$e^{2U} = e^{2\tilde{U}} \prod_\alpha \left(1 - c_\alpha^2 e^{2\tilde{U}}\right)^{-1/2}. \quad (2.11)$$

(Note that indeed the function Y vanishes in the degenerate $N = 4$ case mentioned above, for which the determinant of the matrix in (2.4) vanishes.) After performing the necessary integrals, one then finds that K is given by

$$K = \frac{1}{2} \log \frac{(\sigma + \tilde{\sigma} - k)(\sigma + \tilde{\sigma} + k)}{4\sigma \tilde{\sigma}}. \quad (2.12)$$

To see that this is describing the right solution, we note that in isotropic coordinates the standard single-centre multi-charge black hole is given by

$$\begin{aligned} ds^2 = & -\frac{(1 - \frac{\hat{k}}{R})^2}{(1 + \frac{\hat{k}}{R})^2} \prod_\alpha \left(1 + \frac{4\hat{k}R}{(R + \hat{k})^2} \sinh^2 \mu_\alpha\right)^{-1/2} dt^2 \\ & + \frac{1}{16} \left(1 + \frac{\hat{k}}{R}\right)^4 \prod_\alpha \left(1 + \frac{4\hat{k}R}{(R + \hat{k})^2} \sinh^2 \mu_\alpha\right)^{1/2} (d\rho^2 + dy^2 + \rho^2 d\theta^2), \end{aligned} \quad (2.13)$$

where $R^2 = \rho^2 + y^2$. This metric can be recast into the form of the ansatz (2.2), by performing an holomorphic transformation from the complex coordinate $\xi = \rho + iy$ to the complex coordinate $\eta = r + iz$, in a manner analogous to that observed in the previous cases [10]. Guided by a comparison of the terms proportional to $d\theta^2$, it is not hard to see that the required holomorphic transformation is given by

$$\eta = \frac{1}{4} \left(\prod_{\alpha} \cosh \mu_{\alpha} \right)^{1/2} \left(\xi - \frac{\hat{k}^2}{\xi} \right), \quad (2.14)$$

where $\hat{k} = k (\prod_{\alpha} \cosh \mu_{\alpha})^{-1/2}$, and $c_{\alpha} = \tanh \mu_{\alpha}$. Defining also a rescaled time coordinate $\hat{t} = t (\prod_{\alpha} \cosh \mu_{\alpha})^{1/2}$, we find that the metric (2.2) assumes the form (2.13). This has mass m and charges λ_{α} given by

$$m = \hat{k} \sum_{\alpha} \sinh^2 \mu_{\alpha} + 2\hat{k}N, \quad \lambda_{\alpha} = \frac{1}{2}\hat{k} \sinh 2\mu_{\alpha}. \quad (2.15)$$

It is evident from this that extremality is achieved by sending \hat{k} to zero, with one or more of the parameters μ_{α} going to infinity [8], in such a way that the mass and charges remain finite. Thus in terms of the original parameters k and c_{α} in the axially-symmetric multi-scalar solutions, extremality is achieved if one or more of the c_{α} is equal to 1.

Having identified the single centre multi-scalar black hole solution in the axially symmetric coordinate system that we are using here, it is now straightforward to see how to generalise it to a multi-centre solution. Clearly, since the solutions are given in terms of arbitrary harmonic functions, we may simply take a linear superposition of the basic single-centre solutions described above, located at different points z_n along the z axis. For the time being, we shall continue to take all the harmonic functions \tilde{U}_{α} to be equal, and

$$\tilde{U}_{\alpha} = \tilde{U} \equiv \sum_n \frac{1}{2} \log \frac{\sigma_n + \tilde{\sigma}_n - k_n}{\sigma_n + \tilde{\sigma}_n + k_n}, \quad Y = \frac{1}{2}(4 - N) \tilde{U}, \quad (2.16)$$

where $\sigma_n = \sqrt{r^2 + (z - z_n - k_n/2)^2}$ and $\tilde{\sigma} = \sqrt{r^2 + (z - z_n + k_n/2)^2}$. The solution for K can now be shown to be

$$\begin{aligned} K = & \frac{1}{4} \sum_{m,n=1}^N \log \frac{[\sigma_m \tilde{\sigma}_n + (z - z_m - \frac{1}{2}k_m)(z - z_n + \frac{1}{2}k_n) + r^2]}{[\sigma_m \sigma_n + (z - z_m - \frac{1}{2}k_m)(z - z_n - \frac{1}{2}k_n) + r^2]} \\ & + \frac{1}{4} \sum_{m,n=1}^N \log \frac{[\tilde{\sigma}_m \sigma_n + (z - z_m + \frac{1}{2}k_m)(z - z_n - \frac{1}{2}k_n) + r^2]}{[\tilde{\sigma}_m \tilde{\sigma}_n + (z - z_m + \frac{1}{2}k_m)(z - z_n + \frac{1}{2}k_n) + r^2]}, \end{aligned} \quad (2.17)$$

where $\sigma_n^2 = r^2 + (z - z_n - \frac{1}{2}k_n)^2$ and $\tilde{\sigma}_n^2 = r^2 + (z - z_n + \frac{1}{2}k_n)^2$. In the vicinity of each of the points z_n , the solution is equivalent to the single-centre multi-scalar black hole

described previously, and thus the configuration characterised by (2.17) describes a multi-centre multi-scalar black hole solution, with an independent multi-scalar black hole at each of the locations z_n .

The global structure of such non-extremal multi-black hole solutions was discussed in the simpler cases of Einstein and Einstein-Maxwell solutions in [17, 18]. As one might expect, if a solution with a finite number of centres is considered, or indeed any configuration that is not of a simple periodic nature, then the solution suffers from singularities. This is because in such cases there is no balance of forces on each of the black holes, and thus the system can remain in static equilibrium only if there are “struts” linking the black holes, which supply the necessary additional forces necessary to hold the system in place. These struts manifest themselves in the form of conical curvature singularities on the z axis in the gaps between the black holes [17, 18]. It is clear, however, that if the black holes are arrayed in a simple periodic fashion along the z axis, the net force on each black hole will be zero, and thus no struts are necessary to maintain the static equilibrium. Indeed, under precisely this circumstance, the conical singularities disappear. In a similar manner, we find in the present case of the multi-scalar non-extremal black hole solutions that the metric is free of conical curvature singularities if the “Newtonian masses” k_n are chosen equal, the locations z_n are taken to be of the periodic form $z_n = z_0 + n b$, and the index n is allowed to range over the entire set of integers, positive, negative, and zero. If the spacing b is chosen to be sufficiently small, then seen at a large distance from the z axis, the solution will then limit to a single-centre multi-scalar black hole in the remaining three directions orthogonal to z .

The situation described above may be viewed as a vertical dimensional reduction of multi-scalar non-extremal black holes from four to three dimensions. In the continuum limit where b is very small, one can show [9, 10] that the expressions (2.16) for \tilde{U}_α become the z -independent function $\beta \log r$, where $\beta = k/b$ and b is the inter-black-hole spacing. Substitution into the equations for K in (2.8) then implies that $K = \beta^2 \log r$. From (2.5) we can solve for U , giving

$$e^{2U} = r^{2\beta} \prod_{\alpha} (1 - c_{\alpha}^2 r^{2\beta})^{-1/2} . \quad (2.18)$$

Thus in this continuum limit, the four-dimensional metric ds_4^2 given by (2.2) becomes independent of z , allowing us to perform a Kaluza-Klein reduction to $D = 3$, with

$$ds_4^2 = e^{\varphi} ds_3^2 + e^{-\varphi} dz^2 , \quad (2.19)$$

where φ here denotes the Kaluza-Klein scalar that comes from the dimensional reduction of the 4-dimensional metric. Note that here, and in all other discussions of dimensional

reduction in this paper, we are working always with the Einstein-frame forms for the metrics. The dimensionally-reduced multi-charge black hole metric in $D = 3$ is therefore given by

$$ds_3^2 = -r^{2\beta^2} dt^2 + r^{2\beta^2-4\beta} \prod_{\alpha} (1 - c_{\alpha}^2 r^{2\beta}) (r^{2\beta^2} dr^2 + r^2 d\theta^2) . \quad (2.20)$$

This is, in general, not quite the “usual” kind of N -charge black hole solution in $D = 3$. The reason for this is that usually a solution with N independent charges would involve the excitation of N independent dilatonic scalars. This is indeed the case in our original $D = 4$ solutions (except in the degenerate case $N = 4$, where only three independent scalars are excited.) However upon reduction to $D = 3$ a new scalar, namely the Kaluza-Klein scalar φ above, is also excited. The condition for a normal N -charge, N -scalar solution is that all projections of the full set of $(11 - D)$ dilatonic scalars orthogonal to the N dilaton vectors \vec{a}_{α} in D dimensions should be zero. In the present context, it is not hard to see that this would imply that we should have $\sum_{\alpha} \varphi_{\alpha} - (N - 4)\varphi = 0$. Comparing (2.2) with (2.19), we see that $\varphi = 2U - 2K$, and thus it follows, after substituting in our solutions for U , Y and K in the continuum limit, that the condition for the additional scalar degree of freedom to be unexcited in $D = 3$ is that $\beta = 1$. This implies that the separation b between the black holes is equal to the length k of the Newtonian rods. In fact the Newtonian potential now describes the gravitational field of a single rod of infinite length, and mass density $\frac{1}{2}$. Not surprisingly, the metric in this case is rather pathological, and in fact describes a continuous circle of $D = 3$ black holes in the extremal limit, rather than a single black hole. (This was discussed in detail for the single-charge case in [10].) Thus it would seem that one should really view the more generic solutions with $\beta \neq 1$, and hence with one more independent scalar excitation than normal, as the more appropriate generalisations of black-hole solitons to $D = 3$. In particular, we should consider the case where $\beta < 1$, so that the separation between the black holes in $D = 4$ is less than their horizon radius.

Note that more general multi-centre solutions are also possible. Since the most general solutions that we have constructed allow independent harmonic functions for each kind of scalar field, we may allow the various different kinds of charge to reside at different sets of locations along the z axis. By choosing periodic arrays of centres appropriately, we may thus obtain three-dimensional multi-charge solutions by vertical reduction from $D = 4$.

3 Multi-charge multi-centre black $(D - 4)$ -branes

As we have remarked in the introduction, the existence of special classes of static multi-centre non-extremal p -branes, namely those with a simple periodicity along an axis, plane

or hyperplane, is guaranteed on general grounds since there will be a net balance of forces on each individual p -brane. What is not guaranteed, however, is that there should exist such multi-centre solutions for arbitrarily located sets of non-extremal p -branes. Nevertheless, in the case of black holes in four dimensions, it turns out that axially-symmetric solutions with black holes disposed arbitrarily along an axis do exist. The associated imbalance of forces in general is counterbalanced by the occurrence of “struts” that are associated with conical-type delta-function curvature singularities along the axis between the black holes. In a case where the transverse space has dimension greater than 3, for example black holes in five dimensions, it is not at all clear that analogous axially-symmetric solutions for arbitrarily disposed sets of centres will exist; in particular, the notion of conical singularities with curvature zero except on submanifolds no longer exists. Thus it is quite likely that a description of multi-centre non-extremal p -branes in terms of arbitrary axially-symmetric harmonic functions is no longer possible when the transverse space has dimension greater than 3, although there should certainly exist simply-periodic solutions. The equations of motion for such systems were given in [10], and preliminary investigations indeed seemed to confirm that finding solutions in terms of harmonic functions would be problematical.

If, therefore, we restrict attention to cases where the transverse space has dimension 3, then this implies that we should consider $(D - 4)$ -branes in D dimensions. Black holes in $D = 4$ are thus the simplest case. The form of the axially-symmetric metrics for $(D - 4)$ -branes is like (2.2), but with the addition of p spatial world-volume dimensions:

$$ds^2 = -e^{2U} dt^2 + e^{2A} dx^i dx^i + e^{2K-2U} (dr^2 + dz^2) + e^{2B} r^2 d\theta^2, \quad (3.1)$$

where U , K , A and B are all functions of r and z . The curvature for such metrics was calculated in [10], and from this it is a straightforward matter to obtain the equations of motion for the system with multiple scalars and field strengths described by (2.1), now taken to be in D dimensions. However in practice, a simpler way to construct the required solutions in D dimensions is to make use of the fact that they can be dimensionally reduced to $D = 4$ by compactifying the additional p world-volume dimensions, when they will become black-hole solutions of the kind we have already constructed in section 2. Thus we may obtain the multi-charge multi-centre non-extremal p -branes by inverting the process of Kaluza-Klein dimensional reduction. This oxidation method was used in [10] in order to construct such solutions in the single-charge case.

In order to discuss the oxidation of the four-dimensional black hole solutions, it is useful to consider the step in the Kaluza-Klein reduction process in which we pass from D

dimensions to $(D - 1)$ dimensions. In D dimensions, the dilaton vectors $\hat{\vec{a}}_\alpha$ associated with the N field strengths must satisfy

$$\hat{\vec{a}}_\alpha \cdot \hat{\vec{a}}_\beta = 4\delta_{\alpha\beta} - \frac{2(D-3)}{D-2} \quad (3.2)$$

in order that the equations of motion factorise into a solvable form [16]. Upon reduction to $(D - 1)$ dimensions, the dilatons $\hat{\vec{\phi}}$ will be augmented by the scalar φ_{D-1} coming from the Kaluza-Klein reduction of the metric, and we may represent the full set by the vector $\vec{\phi} = (\hat{\vec{\phi}}, \varphi_{D-1})$. The Kaluza-Klein reduction of the kinetic term for the α 'th field strength will yield an additional coupling of the new scalar field, implying that in $(D - 1)$ dimensions the dilaton vector is given by $\vec{a}_\alpha = (\hat{\vec{a}}_\alpha, 2\alpha_{D-1})$, where $\alpha_{D-1} \equiv (2(D-2)(D-3))^{-1/2}$. It follows that the quantities $\varphi_\alpha = \vec{a}_\alpha \cdot \vec{\phi}$ and $\hat{\varphi}_\alpha = \hat{\vec{a}}_\alpha \cdot \hat{\vec{\phi}}$ in $(D - 1)$ and D dimensions, analogous to the ones we defined in $D = 4$ in section 2, are related by $\hat{\varphi}_\alpha = \varphi_\alpha - 2\alpha_{D-1} \varphi_{D-1}$. Since we are wanting to describe the higher-dimensional solutions that arise from oxidation of the lower-dimensional ones, it must also be the case that the additional scalar field that we acquire in the step from D to $(D - 1)$ dimensions must not lead to any additional independent scalar field in the lower dimensional solution. Thus the linear combination of the new scalar φ_{D-1} and the old scalars $\hat{\vec{\phi}}$ that is orthogonal to those combinations that are excited in the lower-dimensional solution must vanish. This is the combination that is orthogonal to all the dilaton vectors \vec{a}_α in $(D - 1)$ dimensions, *i.e.* the combination parallel to the vector

$$\vec{n} = \left(\sum_\alpha \hat{\vec{a}}_\alpha, \frac{1}{\alpha_{D-1}} \left(\frac{N(D-3)}{D-2} - 2 \right) \right). \quad (3.3)$$

This implies that the scalars in $(D - 1)$ dimensions must satisfy

$$\sum_\alpha \varphi_\alpha + \frac{1}{\alpha_{D-1}} \left(\frac{N(D-4)}{D-3} - 2 \right) \varphi_{D-1} = 0. \quad (3.4)$$

With the above conditions satisfied, it is now a straightforward matter to implement the oxidation procedure recursively, using the standard relation

$$ds_D^2 = e^{2\alpha_{D-1}\varphi_{D-1}} ds_{D-1}^2 + e^{-2(D-3)\alpha_{D-1}\varphi_{D-1}} dz^2 \quad (3.5)$$

between the D -dimensional and $(D - 1)$ -dimensional metrics. This enables us to express the multi-charge multi-centre black $(D - 4)$ -brane solutions in D dimensions in terms of the quantities U , K and φ_α given in section 2 for the four-dimensional black hole solutions. In terms of these, we find the following D -dimensional metric:

$$ds_D^2 = -e^{2U+(D-4)V} dt^2 + e^{-2V} \sum_{i=1}^p dx^i dx^i + e^{-2U+(D-4)V} \left(e^{2K} (dr^2 + dz^2) + r^2 d\theta^2 \right), \quad (3.6)$$

where

$$V = \frac{1}{(D-2)(4-N)} \sum_{\alpha} \varphi_{\alpha} . \quad (3.7)$$

This enables us to describe axially-symmetric multi-centre configurations of non-extremal multi-charge $(D-4)$ -branes in D dimensions.

4 Oxidation to $D = 11$

Having obtained 4-dimensional multi-charge multi-centre non-extremal black holes from the bosonic Lagrangian (2.1) in section 2, we now study how these solutions are embedded in the 4-dimensional maximal supergravity theory. Since the maximal supergravity in $D = 4$ is nothing but a consistent Kaluza-Klein dimensional reduction of 11-dimensional supergravity on the 7-torus, we may oxidise these solutions back to $D = 11$, thus giving an 11-dimensional interpretation of the solutions that we have obtained.

There are a total of twenty-eight 2-form field strengths in 4-dimensional maximal supergravity, each of which can be used to construct either an electrically-charged or a magnetically-charged black hole. These 56 black hole solutions form [19] a multiplet under the E_7 group of the supergravity theory [20, 21]. To be more precise, they form a 56-dimensional multiplet under the Weyl group of E_7 [22]. As we mentioned in section 2, when the dilaton vectors for the 2-form field strengths satisfy the dot products (2.4), there can exist a consistent truncation of the bosonic Lagrangian to (2.1). (It should be remembered that not all sets of field strengths that have this property will lead to a consistent truncation, but at least one member of their Weyl multiplet will [22].) The maximal number of participating 2-form field strengths in the Lagrangian (2.1) in $D = 4$ that can be obtained by consistent truncation of the maximal supergravity theory is $N_{\max} = 4$. As was shown in [22], the dimension of the duality multiplet of this $N = 4$ solution is 630. In other words, there are 630 choices of sets of four 2-form field strengths whose dilaton vectors satisfy the dot products (2.4). In this paper, we shall consider just one example. We are particularly interested in the case where the 2-form field strengths come directly from the dimensional reduction of the 4-form field strength in $D = 11$, rather than those coming from the metric. For these p -brane solutions, the oxidation procedure will give rise to p -branes or intersecting p -branes in all higher dimensions. By contrast, for the solutions involving field strengths coming from the metric, the oxidation would give rise to boosts or topological twists in the metric in higher dimensions.

We follow the notation of [14] for dimensional reduction of 11-dimensional supergravity

on the 7-torus, which was performed by the iteration of a one-step reduction procedure. The 4-form field strength gives rise to seven 3-form field strengths $F_3^{(i)}$, which are dual to 1-forms; twenty-one 2-forms $F_2^{(ij)}$ and thirty-five 1-forms $F_1^{(ijk)}$. The dilaton vectors for these field strengths are denoted by \vec{a}_i , \vec{a}_{ij} and \vec{a}_{ijk} respectively, where the indices i, j and k run over the seven internal compactified coordinates. The explicit forms of these vectors can be found in [14]. There are a further seven 2-forms $\mathcal{F}_2^{(i)}$ and twenty-one 1-forms $\mathcal{F}_1^{(ij)}$ ($i < j$) coming from the metric. In this paper, we shall consider a 4-charge black hole whose charges are carried by the field strengths $F_2^{(12)}$, $F_2^{(34)}$, $*F_2^{(13)}$ and $*F_2^{(24)}$, and corresponding dilaton vectors $\vec{c}_1 = \vec{a}_{12}$, $\vec{c}_2 = \vec{a}_{34}$, $\vec{c}_3 = -\vec{a}_{13}$ and $\vec{c}_4 = -\vec{a}_{24}$. From their expressions given in [14], it is easy to verify that these dilaton vectors satisfy (2.4), and also that the bosonic Lagrangian can be consistently truncated to (2.1), with the standard electric or magnetic ansatz for the field strengths. If we consider an electrically-charged solution, the Hodge duals for the field strengths $*F_2^{(13)}$ and $*F_2^{(24)}$ imply that in terms of the original fields $F_2^{(13)}$ and $F_2^{(24)}$, these field strengths carry magnetic charges; if we consider instead a magnetic solution, these field strengths will describe electric charges. It is this Hodge dualisation that is responsible for the dilaton vectors \vec{c}_3 and \vec{c}_4 having the minus signs that we indicated above.

The 4-dimensional 4-charge multi-centre non-extremal black hole solutions were obtained in section 2. It is straightforward to oxidise the solutions back to $D = 11$, by applying (3.5) iteratively. Defining $H_\alpha = e^{-\tilde{U}_\alpha} - c_\alpha^2 e^{\tilde{U}_\alpha}$, we find that the metric of the corresponding 11-dimension solution is given by

$$\begin{aligned}
ds_{11}^2 = & \left(\frac{H_3 H_4}{H_1 H_2} \right)^{\frac{1}{6}} ds_4^2 + \left(\frac{H_2 H_3^2}{H_1^2 H_4} \right)^{\frac{1}{3}} dz_1^2 + \left(\frac{H_2 H_4^2}{H_1^2 H_3} \right)^{\frac{1}{3}} dz_2^2 + \left(\frac{H_1 H_3^2}{H_2^2 H_4} \right)^{\frac{1}{3}} dz_3^2 \\
& + \left(\frac{H_1 H_4^2}{H_2^2 H_3} \right)^{\frac{1}{3}} dz_4^2 + \left(\frac{H_1 H_2^2}{H_3^2 H_4} \right)^{\frac{1}{3}} (dz_5^2 + dz_6^2 + dz_7^2), \tag{4.1}
\end{aligned}$$

where ds_4^2 is the 4-dimensional metric obtained in section 2. Note that if all $H_\alpha = 1$ except for H_1 , the coordinates (z_1, z_2) become the world-volume spatial coordinates of a membrane solution in $D = 11$. Since there are translational symmetries of the remaining five compactifying coordinates as well, it implies that the metric (4.1) then describes a 5-plane of uniformly distributed non-extremal membranes. If instead only $H_2 \neq 1$, then the metric describes a 5-plane of non-extremal membranes with (z_3, z_4) as the world-volume spatial coordinates. The story is slightly different with the other two cases. If only $H_3 \neq 1$, then the solution describes a 2-planes of non-extremal 5-branes with $(z_2, z_4, z_5, z_6, z_7)$ as the world-volume spatial coordinates; if only $H_4 \neq 1$, it describes a 2-plane of non-extremal 5-branes with $(z_1, z_3, z_5, z_6, z_7)$ as the world-volume spatial coordinates. Thus in general

the metric (4.1) describes the intersection of two membranes and two 5-branes in eleven dimensions. Oxidations of isotropic non-extremal black holes were considered in [23]. In this paper, our solutions depend on a set of harmonic functions \tilde{U}_α , which generalises previous results. In the extremal limit, the functions H_α themselves become harmonic functions on the 3-dimensional space (r, z, θ) , which enables one to interpret 4-dimensional single-centre single-scalar or multi-scalar black hole solutions as bound states of singly-charged black holes [24].

In the above discussion, we considered the electric four-dimensional solutions (with the pre-dualisation of the 3'rd and 4'th field strengths, of course). If we instead consider magnetic solutions, the 11-dimensional oxidation is still given by (4.1), except this time we have $H_\alpha \rightarrow 1/H_\alpha$. Thus in this case, the H_1 and H_2 will instead be associated with 5-branes in $D = 11$, and H_3 and H_4 with membranes. As we mentioned earlier, there are more choices of sets of field strengths that give rise to a consistent truncation of the bosonic Lagrangian to (2.1). For example, we could instead use $F_2^{(12)}$, $F_2^{(34)}$, $F_2^{(56)}$ and $*\mathcal{F}_2^{(7)}$ [14]. The four-dimensional metric is identical to the previous solutions; however, since we are now using a different field-strength configuration, the oxidation to $D = 11$ will be quite different. The involvement of the 2-form $\mathcal{F}_2^{(7)}$ coming from the metric implies a twist or a boost in the 11-dimensional metric. Note that the 4-dimensional 4-charge black holes always involve at least one dualised field strength. In other words, in terms of the original fields, the solutions always involve both electric and magnetic charges. Such solutions were called dyonic solutions of the first type in [14], since each individual field strength nevertheless carries only one type of charge. In section 5.3, we shall consider non-extremal dyonic black hole solutions of the second type, where a single 2-form field strength carries both electric and magnetic charges.

5 Solutions and symmetries

In this section, we shall consider the general question of how to construct solutions to the systems of equations that arise for the axially-symmetric configurations. To begin, we present a procedure for constructing a class of solutions to the single-scalar equations of motion, expressed in terms of two independent harmonic functions. In fact these solutions coincide, in the case $N = 1$, with the general N -scalar solutions presented in the previous section. Then we shall consider a different approach to obtaining these solutions, which exploits an $O(2, 1)$ symmetry of the equations. In a subsequent subsection, we shall then

apply similar considerations to the more complicated case of single-scalar dyonic black hole solutions.

5.1 Single-scalar solutions

We begin by considering the simplest situation, discussed in [10], where there is just a single 2-form field strength and a single scalar field. The four-dimensional Lagrangian is then given by

$$\mathcal{L} = eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}e e^{-a\phi} F^2 . \quad (5.1)$$

Substituting the metric ansatz (2.2) and the ansatz $A = \gamma dt$ for an electric charge, we obtain the equations of motion

$$\begin{aligned} \nabla^2 U &= \frac{1}{4} e^{-a\phi-2U} (\nabla\gamma)^2 , & \nabla^2 \phi &= \frac{1}{2} a e^{-a\phi-2U} (\nabla\gamma)^2 , \\ \nabla^2 \gamma &= \nabla\gamma \cdot \nabla(a\phi + 2U) , \end{aligned} \quad (5.2)$$

together with the equations that determine K in (2.2), namely

$$\begin{aligned} K' &= \frac{1}{4} r e^{-a\phi-2U} (\dot{\gamma}^2 - \gamma'^2) + r (U'^2 - \dot{U}^2) + \frac{1}{4} r (\phi'^2 - \dot{\phi}^2) , \\ \dot{K} &= -\frac{1}{2} r e^{-a\phi-2U} \dot{\gamma} \gamma' + 2r \dot{U} U' + \frac{1}{2} r \dot{\phi} \phi' . \end{aligned} \quad (5.3)$$

In order to solve the equations, we need only consider the system given in (5.2), since the solution for K then follows from (5.3) by quadratures. In (5.2), we do not yet need to assume that the functions U , ϕ and γ are independent of the axial angle θ , and so the Laplacian ∇^2 and the gradient operator ∇ itself can be thought of for now as the ordinary 3-dimensional operators in the flat transverse space. We shall proceed by looking for ways to solve the equations (5.2). It is convenient first to introduce two new functions f and g in place of U and ϕ , defined by

$$f = a\phi + 2U , \quad g = \phi - 2aU . \quad (5.4)$$

In terms of these, the equations of motion (5.2) become

$$\nabla^2 f = \frac{1}{2} \Delta e^{-f} (\nabla\gamma)^2 , \quad \nabla^2 g = 0 , \quad \nabla^2 \gamma = \nabla\gamma \cdot \nabla f . \quad (5.5)$$

One approach to solving (5.5) is to note that the solutions obtained in [10] are such that U , ϕ and γ are all expressed as certain algebraic functions of a single harmonic function \tilde{U} . In fact, from the form of the equations (5.5), we see that g is already an harmonic function, and furthermore that it does not appear at all in the equations of motion for f

and γ . Thus it is natural to try a more general ansatz, in which we assume that f and γ are both algebraic functions of a second, independent, harmonic function ψ , *i.e.* $f = f(\psi)$ and $\gamma = \gamma(\psi)$. Since $\nabla^2 \psi$ is assumed to vanish, it follows that the equations of motion for f and γ in (5.5) become

$$\frac{d^2 f}{d\psi^2} = \frac{1}{2} \Delta e^{-f} \left(\frac{d\gamma}{d\psi} \right)^2, \quad \frac{d^2 \gamma}{d\psi^2} = \frac{df}{d\psi} \frac{d\gamma}{d\psi}. \quad (5.6)$$

The latter can be integrated once to give $d\gamma/d\psi = (c/\sqrt{\Delta}) e^f$, where c is a constant whose normalisation is chosen for later convenience. Substituting this into the first equation in (5.6) gives the Liouville equation

$$\frac{d^2 f}{d\psi^2} = \frac{1}{2} c^2 e^f. \quad (5.7)$$

This has the solution

$$e^{-\frac{1}{2}f} = e^{-\psi} - c^2 e^{\psi}, \quad (5.8)$$

where the constants of integration are absorbed into the freedom to scale and shift the arbitrary harmonic function ψ by constants. Using the Liouville equation (5.7) we can integrate the equation for γ to give $\gamma = 2/(\sqrt{\Delta}c) df/d\psi + \text{const.}$ Since γ is a potential for the electric field we can choose the constant arbitrarily. Making a convenient choice, we find that the solution is

$$\gamma = \frac{2c}{\sqrt{\Delta}} e^{2\psi} (1 - c^2 e^{2\psi})^{-1}. \quad (5.9)$$

Thus the entire solution for U , ϕ and γ is given in terms of two independent arbitrary harmonic functions g and ψ . The equations for K in (5.3) reduce to

$$\begin{aligned} K' &= \frac{r}{4\Delta} (4\dot{\psi}^2 - 4\dot{\psi}^2 + g'^2 - \dot{g}^2), \\ \dot{K} &= \frac{r}{2\Delta} (\dot{g} g' + 4\dot{\psi} \psi'). \end{aligned} \quad (5.10)$$

The previous solutions obtained in [10] correspond to the case where the two harmonic functions g and ψ are chosen to be proportional to a single harmonic function \tilde{U} , with $\psi = \tilde{U}$ and $g = -2a\tilde{U}$. There are other special cases too, such as $\psi = 0$ which implies that γ is simply a constant, and hence there is no electric charge. Since f is also constant in this case, it implies that U and ϕ are both harmonic, and are proportional to the remaining harmonic function g . Another special case is $g = 0$, implying that ϕ and U are related, $\phi = 2aU$, and given in terms of the remaining harmonic function ψ . Finally, we note that if $c = 0$ the electric charge again vanishes, and ϕ and U become independent harmonic functions. The special case when ϕ also vanishes corresponds to a solution of the pure Einstein equations.

5.2 Solution-generating symmetry of the single-scalar system

As another approach to solving the equations of motion, it is useful to consider the symmetries of the system of equations (5.5). Since the equation for g is independent of the others, we may focus principally on the equations of motion for f and γ . It is easy to see that these can be derived from the Lagrangian

$$L = (\nabla f)^2 - \Delta e^{-f} (\nabla \gamma)^2 . \quad (5.11)$$

Let us now introduce the three fields X , Y and Z , defined by $X + Y = 2e^{-f/2}$, $X - Y = 2e^{f/2} - \frac{1}{2}\Delta\gamma^2 e^{-f/2}$ and $Z = \sqrt{\Delta}\gamma e^{-f/2}$. In terms of these, we have

$$X^2 - Y^2 + Z^2 = 4 , \quad (5.12)$$

$$L = -(\nabla X)^2 + (\nabla Y)^2 - (\nabla Z)^2 . \quad (5.13)$$

The constraint (5.12) and the Lagrangian (5.13) have a manifest $O(2, 1)$ symmetry. In terms of the original fields f and γ , this symmetry indeed gives an invariance of the Lagrangian (5.11), but owing to the choice of parameterisation we should restrict to transformations that do not reverse the sign of $e^{f/2}$ (*i.e.* we should restrict to transformations where $X + Y$ is non-negative). Thus the allowed symmetry transformations are of the form $O(2, 1)/J$, where J is the antipodal map $(X, Y, Z) \rightarrow (-X, -Y, -Z)$. Since J is central in $O(2, 1)$, it follows that $O(2, 1)/J$ is a group. If an $O(2, 1)$ transformation makes $X + Y$ negative, then J is used to restore its positivity.¹

The symmetry group $O(2, 1)/J$ has two disconnected components, one of which is $SO(2, 1)$ and the other is reached by performing the discrete “time-reversal” transformation $Y \rightarrow -Y$. Since $SL(2, R)$ is homomorphic to $SO(2, 1)$, we can also express the $SO(2, 1)$ subgroup in a manner analogous to a fractional linear transformation. To see this, let us introduce the field $\chi = -i\gamma$. Temporarily treating χ as a real field, we may rewrite the Lagrangian (5.11) as

$$L = -\frac{\nabla \tau \cdot \nabla \bar{\tau}}{(\tau - \bar{\tau})^2} , \quad (5.14)$$

where we have defined

$$\tau = \frac{1}{2}\sqrt{\Delta}\chi + i e^{\frac{1}{2}f} . \quad (5.15)$$

This is invariant under the fractional linear $SL(2, \mathbf{R})$ transformations

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} , \quad (5.16)$$

¹We are grateful to G.W. Gibbons for discussions about the symmetry group of the Lagrangian (5.11).

where $ad - bc = 1$. Rewriting this transformation in terms of its action on the fields γ and f , which involves also sending $b \rightarrow -ib$ and $c \rightarrow ic$ in order to make the transformation real again, we find

$$\begin{aligned}\gamma &\longrightarrow \frac{2((\frac{1}{2}a\sqrt{\Delta}\gamma + b)(\frac{1}{2}c\sqrt{\Delta}\gamma + d) - ac e^f)}{\sqrt{\Delta}((\frac{1}{2}c\sqrt{\Delta}\gamma + d)^2 - c^2 e^f)} , \\ e^{\frac{1}{2}f} &\longrightarrow \frac{e^{\frac{1}{2}f}}{(\frac{1}{2}c\sqrt{\Delta}\gamma + d)^2 - c^2 e^f} ,\end{aligned}\tag{5.17}$$

where again $ad - bc = 1$.

The $O(2,1)/J$ symmetry provides a powerful way of constructing new solutions from old ones. In particular, the “time-reversal” symmetry $Y \rightarrow -Y$ of (5.12) and (5.13) implies that

$$\gamma \longrightarrow -\frac{\gamma}{ef - \frac{1}{4}\Delta\gamma^2} , \quad e^{\frac{1}{2}f} \longrightarrow \frac{e^{\frac{1}{2}f}}{ef - \frac{1}{4}\Delta\gamma^2} .\tag{5.18}$$

(Note that $Y \rightarrow -Y$ is contained in $O(2,1)/J$, but is not in the subgroup $SO(2,1)$ that is connected to the identity.) Let us apply this to the simple solution

$$\gamma = -\frac{2}{\sqrt{\Delta}}c , \quad f = -2\psi ,\tag{5.19}$$

where c is a constant and ψ is an harmonic function. It is manifest that (5.19) satisfies the equations of motion given in (5.5). In fact, it describes an uncharged solution, *i.e.* a solution of the pure Einstein-Dilaton equations. Under the transformation (5.18), it becomes precisely the charged Einstein-Maxwell-Dilaton solution given in (5.9) and (5.8). Thus the discrete $Y \rightarrow -Y$ transformation in the $O(2,1)/J$ symmetry of the equations of motion maps between uncharged and charged solutions. We note also that the expressions for K' and \dot{K} given in (5.3) can be cast into the form

$$\begin{aligned}K' &= -\frac{4r}{\Delta} \frac{(\tau' \bar{\tau}' - \dot{\tau} \dot{\bar{\tau}})}{(\tau - \bar{\tau})^2} + \frac{r}{4\Delta} (g'^2 - \dot{g}^2) , \\ \dot{K} &= -\frac{4r}{\Delta} \frac{(\tau' \dot{\bar{\tau}} + \dot{\tau} \bar{\tau}')}{(\tau - \bar{\tau})^2} + \frac{r}{2\Delta} \dot{g} g' .\end{aligned}\tag{5.20}$$

(After writing out these expressions in terms of χ and f , one should again make the replacement $\chi = -i\gamma$.) This shows that K is invariant under the $SL(2, \mathbf{R})$ transformation. In fact, it is also invariant under the entire $O(2,1)/J$ symmetry. In particular, this explains why the expressions for K' and \dot{K} that we obtained in (5.10) for the charged solutions are the same as they would be for uncharged solutions.

By varying the Lagrangian (5.11) with respect to local infinitesimal $SL(2, \mathbf{R})$ transformations, we can read off the Noether currents that generate the symmetry. They are

proportional to

$$\begin{aligned} J_1 &= r e^{-f} \nabla \gamma , \\ J_2 &= r \nabla f - \frac{1}{2} \Delta r e^{-f} \gamma \nabla \gamma , \end{aligned} \quad (5.21)$$

$$J_3 = r \nabla \gamma - r \gamma \nabla f - \frac{1}{4} \Delta r e^{-f} \gamma^2 \nabla \gamma . \quad (5.22)$$

The associated conserved charges, which are independent of r , are obtained by integrating the r -components of these currents over all z . In particular, the first current in (5.21) integrates to give the electric charge carried by the field strength F :

$$\begin{aligned} Q &= \frac{1}{4\pi} \int *F e^{-a\phi} = \frac{1}{4\pi} \int *(d\gamma \wedge dt) e^{-a\phi} , \\ &= \int r e^{-f} \gamma' dz . \end{aligned} \quad (5.23)$$

The multi-scalar equations (2.6) can be analysed in a similar manner. In this case, we find that the equations for Φ_α and γ_α can be obtained from the Lagrangian

$$L = \sum_{\alpha=1}^N \left((\nabla \Phi_\alpha)^2 - 4e^{-\Phi_\alpha} (\nabla \gamma_\alpha)^2 \right) . \quad (5.24)$$

Clearly this has an $(O(2,1)/J)^N$ symmetry, which again may be used to obtain new solutions from old ones, in the same way as we described for the single-scalar case above. In particular, we could begin with a pure Einstein-Dilaton solution with no electric charges at all, and use the N independent $O(2,1)/J$ symmetries to turn on the N charges. For example, by starting from the uncharged solution where Y and Φ_α are independent harmonic functions, and the functions γ_α are appropriately chosen constants, we can reproduce the general class of charged solutions given in (2.8).

5.3 Dyonic solutions

Now let us consider the situation when the 2-form field strength carries both electric and magnetic charge, in which case F takes the form

$$F = (\gamma' dr + \dot{\gamma} dz) \wedge dt + r e^{a\phi-2U} (\tilde{\gamma}' dz - \dot{\tilde{\gamma}} dr) \wedge d\theta , \quad (5.25)$$

and the equations of motion become

$$\begin{aligned} \nabla^2 U &= \frac{1}{4} e^{-a\phi-2U} (\nabla \gamma)^2 + \frac{1}{4} e^{a\phi-2U} (\nabla \tilde{\gamma})^2 , \\ \nabla^2 \phi &= \frac{1}{2} a e^{-a\phi-2U} (\nabla \gamma)^2 - \frac{1}{2} a e^{a\phi-2U} (\nabla \tilde{\gamma})^2 , \\ \nabla \cdot (e^{-a\phi-2U} \nabla \gamma) &= 0 , \\ \nabla \cdot (e^{a\phi-2U} \nabla \tilde{\gamma}) &= 0 , \end{aligned} \quad (5.26)$$

$$(5.27)$$

together with the equations that determine K in (2.2), namely

$$\begin{aligned} K' &= \frac{1}{4}r e^{-a\phi-2U} (\dot{\gamma}^2 - \gamma'^2) + \frac{1}{4}r e^{a\phi-2U} (\dot{\tilde{\gamma}}^2 - \tilde{\gamma}'^2) + r(U'^2 - \dot{U}^2) + \frac{1}{4}r(\phi'^2 - \dot{\phi}^2) , \\ \dot{K} &= -\frac{1}{2}r e^{-a\phi-2U} \dot{\gamma} \gamma' - \frac{1}{2}r e^{a\phi-2U} \dot{\tilde{\gamma}} \tilde{\gamma}' + 2r \dot{U} U' + \frac{1}{2}r \dot{\phi} \phi' . \end{aligned} \quad (5.28)$$

Defining two new functions q_1 and q_2 in terms of ϕ and U , given by

$$a q_1 = \phi + 2a U , \quad a q_2 = -\phi + 2a U , \quad (5.29)$$

the equations of motion become

$$\begin{aligned} \nabla^2 q_1 &= e^{-\alpha q_1 - (1-\alpha)q_2} (\nabla \gamma)^2 , \\ \nabla^2 q_2 &= e^{-\alpha q_2 - (1-\alpha)q_1} (\nabla \tilde{\gamma})^2 , \\ \nabla \cdot (e^{-\alpha q_1 - (1-\alpha)q_2} \nabla \gamma) &= 0 , \end{aligned} \quad (5.30)$$

$$\nabla \cdot (e^{-\alpha q_2 - (1-\alpha)q_1} \nabla \tilde{\gamma}) = 0 , \quad (5.31)$$

where the constant α is given by $\alpha = \frac{1}{2}(1 + a^2) = \frac{1}{2}\Delta$. For general values of a it seems not to be possible to find solutions to these equations other than the rather trivial case where $q_1 = q_2$ and $\gamma = \tilde{\gamma}$, for which the equations reduce to

$$\nabla^2 q_1 = e^{-q_1} (\nabla \gamma)^2 , \quad \nabla \cdot (e^{-q_1} \nabla \gamma) = 0 . \quad (5.32)$$

This system can be solved by analogous methods to those that we used in section 5.1, by writing q_1 and γ as functions of a single harmonic function ψ . This reduces it to the Liouville equation. Note that in this special solution, the dilaton ϕ is zero.

There are two special values of a for which more general solutions can be given. Firstly, if we take $a = 1$, which implies that $\alpha = 1$, we see that the equations (5.30) decouple. They can be solved by taking q_1 and γ to be functions of an harmonic function ψ_1 , and q_2 and $\tilde{\gamma}$ to be functions of an independent harmonic function ψ_2 , reducing the system to two independent Liouville equations, with the solutions

$$\begin{aligned} e^{-\frac{1}{2}q_1} &= e^{-\psi_1} - c_1^2 e^{\psi_1} , & \gamma &= \sqrt{2}c_1 e^{2\psi_1} (1 - c_1^2 e^{2\psi_1})^{-1} , \\ e^{-\frac{1}{2}q_2} &= e^{-\psi_2} - c_2^2 e^{\psi_2} , & \tilde{\gamma} &= \sqrt{2}c_2 e^{2\psi_2} (1 - c_2^2 e^{2\psi_2})^{-1} , \end{aligned} \quad (5.33)$$

where c_1 and c_2 are arbitrary constants. The other special case is $a = \sqrt{3}$, which implies $\alpha = 2$. Now, if we take q_1 , q_2 , γ and $\tilde{\gamma}$ all to be functions of a single harmonic function ψ , the equations reduce to the $SL(3, R)$ Toda equations

$$\frac{d^2 q_1}{d\psi^2} = e^{2q_1 - q_2} , \quad \frac{d^2 q_2}{d\psi^2} = e^{2q_2 - q_1} , \quad (5.34)$$

where we have absorbed two constants of integration into rescalings of γ and $\tilde{\gamma}$ together with shift transformations of q_1 and q_2 . The solutions for γ and $\tilde{\gamma}$ are $\gamma = dq_1/d\psi$ and $\tilde{\gamma} = dq_2/d\psi$. The general solution to the Toda equations (5.34) can be written as

$$\begin{aligned} e^{-q_1} &= \frac{c_1 e^{\mu_1 \psi}}{\nu_1(\nu_1 - \nu_2)} + \frac{c_2 e^{\mu_2 \psi}}{\nu_2(\nu_1 - \nu_2)} + \frac{e^{-(\mu_1 + \mu_2)\psi}}{c_1 c_2 \nu_1 \nu_2} , \\ e^{-q_2} &= \frac{e^{-\mu_1 \psi}}{c_1 \nu_1(\nu_1 - \nu_2)} + \frac{e^{-\mu_2 \psi}}{c_2 \nu_2(\nu_1 - \nu_2)} + \frac{c_1 c_2 e^{(\mu_1 + \mu_2)\psi}}{\nu_1 \nu_2} , \end{aligned} \quad (5.35)$$

where μ_1 , μ_2 , c_1 and c_2 are arbitrary constants, $\nu_1 \equiv 2\mu_1 + \mu_2$ and $\nu_2 \equiv 2\mu_2 + \mu_1$. This case where $a = \sqrt{3}$ is actually realised in maximal supergravity in $D = 4$.

We can examine the symmetries of the dyonic system of equations in a manner analogous to that discussed in section 5.2 for the purely electric case. It turns out that for arbitrary a the system possesses only the following symmetries:

$$\begin{aligned} \gamma &\rightarrow e^\lambda \gamma , & \tilde{\gamma} &\rightarrow e^{\tilde{\lambda}} \tilde{\gamma} , \\ \phi &\rightarrow \phi + \frac{1}{2}(\lambda - \tilde{\lambda}) , & U &\rightarrow U + \frac{1}{4}(\lambda + \tilde{\lambda}) , \end{aligned} \quad (5.36)$$

where λ and $\tilde{\lambda}$ are constants. However, when $a = 1$ (and for no other values), the symmetry is enlarged to $(O(2,1)/J)^2$. This is not surprising, since at this value of a we saw that the system decouples into two independent Liouville equations, which, as we saw previously, will each have an $O(2,1)/J$ symmetry. In this $a = 1$ case the symmetry can be used to produce new solutions from old ones.

6 Conclusions

In this paper, we have studied classes of non-extremal black hole solutions in 4-dimensional supergravity. Owing to special properties of four dimensions, it turns out that solutions can be given, for arbitrary distributions of black holes along an axis, in terms of harmonic functions. The forces between the individual black holes are balanced by “struts” located between them, on which the curvature has conical singularities. These struts, and their associated singularities, disappear in the cases of principal interest to us, namely for simply-periodic arrays where there is a net balance of forces on each black hole. In an appropriate dense limit, such a periodic array can be used to describe a vertical dimensional reduction to black hole solutions in three dimensions.

The cases that we were able to solve included multi-charge black holes in four dimensions, where several different field strengths and dilatonic scalars in the four-dimensional theory are involved. These solutions can also be diagonally oxidised in a straightforward fashion, to

give multi-charge non-extremal $(D-4)$ -branes in $D \geq 5$ dimensions. One can also investigate the oxidation of such solutions all the way back to the original $D = 11$ supergravity from which the four-dimensional theory can be viewed as originating.

We also examined the special case of the dyonic black hole in four dimensions. This involves a single field strength, carrying both electric and magnetic charges. We were able to construct a particular class of multi-centre solutions, describing arbitrary distributions of dyonic black holes along an axis, where each individual black hole carries the same ratio of electric to magnetic charge. This is sufficient for the purposes of performing a vertical reduction to $D = 3$. It would also be very interesting if one could obtain more general solutions in which the individual electric and magnetic constituent charges could be separated in different centres. This “pulling apart” of the dyon would enable its stability properties to be investigated in detail. This would be of great interest because even in the extremal limit, the dyonic black hole has non-vanishing binding energy; it is negative, implying that there should be a tendency for the electric and magnetic charges to separate into individual electric and magnetic black holes [13]. Unfortunately, we were unable to solve the equations in this case, although it may nevertheless be that they would be tractable given suitable techniques for solving them.

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